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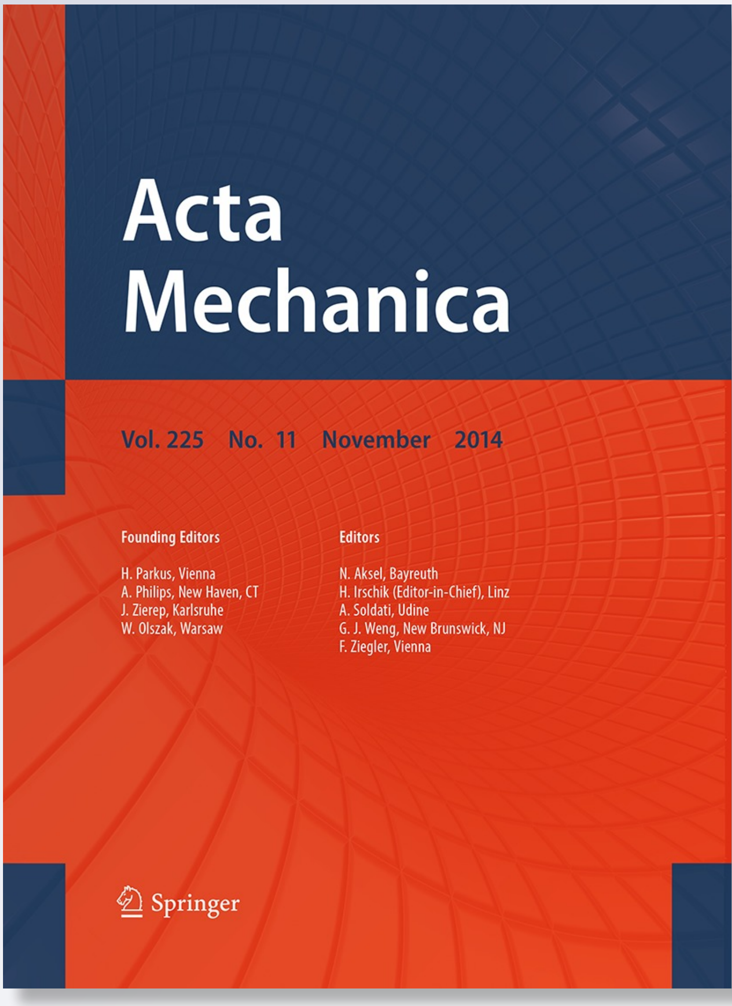

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Abstract We consider an axisymmetric Stokes flow in an infinite right circular cone, which has a source of momentum (a Stokeslet) on its axis. It produces an infinite sequence of eddies in the conical flow region. A boundary problem for a stream function is solved. The picture of the streamlines is obtained. We investigate an eddy structure of the flow. The results can be used for constructing nanoreactors while carrying out chemical reactions in strictly localized nanosized spatial regions.

Keywords Stokes flow · Stokeslet · Cone · Eddy

1 Introduction

Currently, an investigation of a fluid flow in nanotubes and other nanostructures has become one of the topical issues of nanohydrodynamics [1,2]. Such kind of a flow takes place under small Reynolds number; therefore, the inertia terms can be neglected and slow creeping flows can be considered [3–6]. They are described by Stokes equations. Let us consider the following model object: a cone, which has a unit force at some point on its axis, particularly, a Stokeslet. The last is an abstraction, which corresponds to an infinitely small area and drives a fluid. Earlier the solution of a problem of an axisymmetric flow driven by an infinite circular cone has been derived [7]. Steady axisymmetric converging flow under nonzero values of the Reynolds number has been also investigated by Ackerberg [8]. He found the form of a stream function as a solution of the Navier–Stokes equation as well as its asymptotic expansion.

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Viscous creeping flow near a vertex of a cone has been studied by Wakiya [9]. He showed the existence of an infinite set of eddies near the apex if the half-angle of a cone is <80.9°. The author of [10] considered axisymmetric Stokes flow for the same geometry due to rotation of a small sphere with a given angular velocity. The classical review of these and other results concerning Stokeslets operating in a fluid has been given in [11]. The flow in a corner due to a Stokeslet is considered in [12]. Liu and Joseph [13] developed a theory, which led to a new set of eigenfunctions of the Stokes flow in infinite conical trenches, which described axisymmetric flow near the vertex. Later, the numerical solution of the problem of a closed cylinder or a cone with a fluid inside, which is driven by a sphere of a small radius, has been derived in [14]. Creeping flow and emerging eddies were got in [15] in the case when there is a nonzero velocity at the boundary within a ring $0 < a_1 < r < a_2 < \infty$, r being the distance from the apex of a cone. Moreover, Ref. [16] is devoted to the Stokes flow between two coaxial cones with a source at the apex of an outer cone. The flow inside a rolling nanocone is considered in [17].

A flow caused by a point force, Stokeslet, near a plane wall was considered by [18]. In [19], the flow induced by a Stokeslet in a spherical cavity was studied. Stokes flow due to a Stokeslet in a domain between two parallel flat plates was studied in [20], and in a pipe in [21,22]. An infinite Stokeslet array has been considered in [23]. From the mathematical point of view, a Stokeslet can be rigorously introduced in the framework of the theory of self-adjoint extensions of symmetric operators [24,25] and [26].

Thus, it is of an interest to investigate fluid motion due to a Stokeslet in a cone. In this paper, a cone with a fluid driven by a Stokeslet on its axis is considered under Stokes approximation. The result of this work provides a new method for solving a particular axisymmetric case of a fluid flow caused by a Stokeslet inside a cone.

2 Problem solution

Let us introduce the Cartesian and spherical polar coordinate systems.

Consider fluid that reposes in an unbounded cone (Fig. 1). We add a Stokeslet to the axis of a cone, at point $(x_0, y_0, z_0) = (0, 0, c)$ in the Cartesian coordinate system. A Stokeslet is a unit force, which sets fluid in motion. The Stokeslet is given by the corresponding Oseen tensor, which is the Green function, and determines its velocity field [18]:

$$G_j^k = \frac{1}{8\pi\mu}\left(\frac{\delta_{jk}}{|\mathbf{r}|} + \frac{r_j r_k}{|\mathbf{r}|^3}\right), \quad \mathbf{r} = (x - x_0, y - y_0, z - z_0). \tag{1}$$

The velocity components in directions ρ, θ and ϕ can be found via transformation of physical vector components [27]. We restrict our attention to the axisymmetric case, that is, the components of the Stokeslet's velocity depend only on ρ and θ. Among three possible Stokeslets, only one satisfies the condition $v_{0\phi} = 0$. This is the Stokeslet with components $\mathbf{v}_0 = \left(G_1^3, G_2^3, G_3^3\right)$. Its components in the directions ρ, θ, and ϕ are as follows:

$$v_{0\rho} = \frac{1}{8\pi\mu}\frac{2\cos\theta\left(c^2 + r^2\right) - cr\left(3\cos^2\theta + 1\right)}{\left(c^2 - 2cr\cos\theta + r^2\right)^{3/2}}, \tag{2}$$

$$v_{0\theta} = -\frac{1}{8\pi\mu}\frac{\sin\theta\left(2c^2 - 3cr\cos\theta + r^2\right)}{\left(c^2 - 2cr\cos\theta + r^2\right)^{3/2}}, \quad v_{0\phi} = 0. \tag{3}$$

The Stokes flow of an axisymmetric case is characterized by a stream function $\psi(\rho, \theta)$ [7]. The velocity components in this case are

$$v_\rho = \frac{1}{\rho^2\sin\theta}\frac{\partial\psi}{\partial\theta}, \quad v_\theta = -\frac{1}{\rho\sin\theta}\frac{\partial\psi}{\partial\rho}. \tag{4}$$

It is of an interest for us to find contourlines of the stream function of a fluid moved by the Stokeslet $\mathbf{v}_0$. Let us state the problem. The Stokes equation for an axisymmetric case is as follows:

$$E^2(E^2\psi) = 0, \tag{5}$$

where the operator E^2 is the Stokes operator defined by the following expression:

$$E^2 \equiv \frac{\partial^2}{\partial\rho^2} + \frac{\sin\theta}{\rho^2}\frac{\partial}{\partial\theta}\left(\frac{1}{\sin\theta}\frac{\partial}{\partial\theta}\right). \tag{6}$$

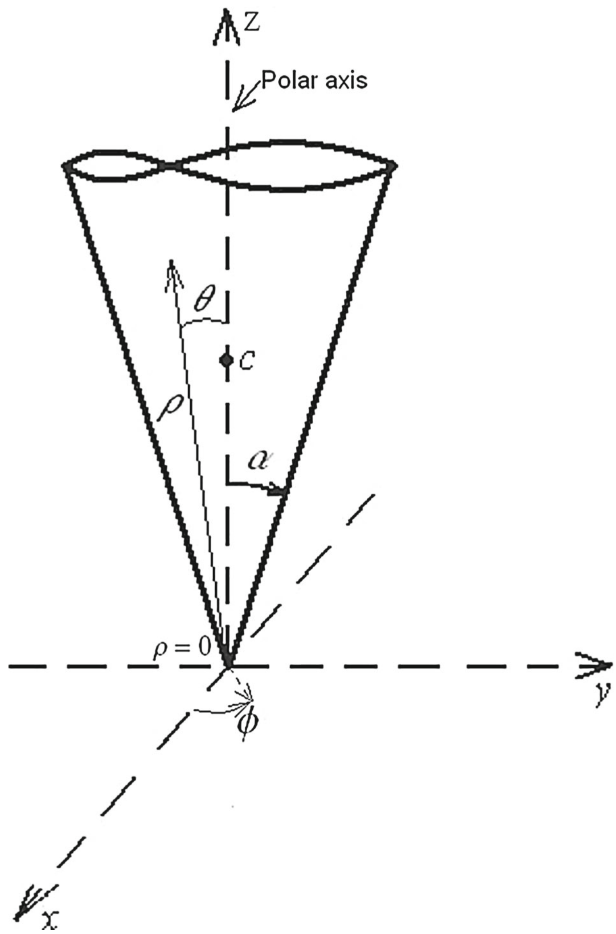

Fig. 1 Cartesian and spherical coordinates for our problem

We add no-slip boundary conditions $v_\rho(\rho,\alpha)=0$ and $v_\theta(\rho,\alpha)=0$.

Let us turn to the Stokeslet $\mathbf{v}_0$ once more. Using (2), (3), (4), and the well-known relation $\frac{\partial}{\partial\theta}\left(\frac{\partial\psi}{\partial\rho}(\rho,\theta)\right)=\frac{\partial}{\partial\rho}\left(\frac{\partial\psi}{\partial\theta}(\rho,\theta)\right)$, we find the stream function ψ_0 of the Stokeslet $\mathbf{v}_0$ with no boundaries:

$$\psi_0(\rho,\theta)=-\frac{\rho^2\sin^2\theta}{\sqrt{\rho^2-2c\rho\cos\theta+c^2}}. \tag{7}$$

The streamline pattern for a flow due to a Stokeslet in an infinite quiescent fluid is very well known (see, e.g., [28]). Particularly, there are no eddies, which appear due to the interaction of the flow with the boundaries. To obtain the stream function for our boundary problem, we represent it in the form $\psi=\psi_0+\tilde{\psi}$. The operator E^2 is linear, so $E^2(E^2\psi)=E^2(E^2\psi_0)+E^2(E^2\tilde{\psi})=0$. The boundary conditions are $\mathbf{v}(\rho,\theta)|_\alpha=(\mathbf{v}_0(\rho,\theta)+\tilde{\mathbf{v}}(\rho,\theta))|_\alpha=0$. One can convince himself that the stream function ψ_0 satisfies the Stokes equation $E^2\left(E^2\psi_0\right)=0$. Knowing the stream function ψ_0 and Stokeslet's velocity components $v_{0\rho}$ and $v_{0\theta}$, we reformulate the problem:

$$E^2\left(E^2\tilde{\psi}\right)=0, \tag{8}$$

$$\tilde{v}_\rho(\rho,\alpha)=-v_{0\rho}(\rho,\alpha),\quad \tilde{v}_\theta(\rho,\alpha)=-v_{0\theta}(\rho,\alpha). \tag{9}$$

The general solution of Eq. (8) is known [7]. As we are looking for a smooth solution, without any singularities on the axis of the cone, the velocity of the fluid and the stream function $\tilde{\psi}$ has the form:

$$\tilde{\psi}(\rho,\theta) = \sum_{n=0}^{\infty} \left(A_n\rho^n + C_n\rho^{n+2}\right) J_n(\cos\theta),$$

$$\tilde{v}_\rho(\rho,\theta) = -\sum_{n=1}^{\infty} \left(A_n\rho^{n-2} + C_n\rho^n\right) P_{n-1}(\cos\theta),$$

$$\tilde{v}_\theta(\rho,\theta) = \sum_{n=0}^{\infty} \left(nA_n\rho^{n-2} + (n+2)C_n\rho^n\right) \frac{J_n(\cos\theta)}{\sin\theta}$$

where $J_n(\zeta)$ are the Gegenbauer functions of the first kind. They are linearly related with the Legendre functions $P_n(\zeta)$:

$$J_n(\zeta) = \frac{P_{n-2}(\zeta) - P_n(\zeta)}{2n-1}, \quad (n \geq 2), \quad J_0(\zeta) = 1, \quad J_1(\zeta) = -\zeta.$$

We make an orthogonal decomposition of the velocity components at the boundary by Laguerre's polynomials L_k. These polynomials form a complete orthogonal basis for $L^2[0,\infty)$, for which the weight function is $e^{-\rho}$. It is known that if function $f(\rho)$ is piecewise smooth on an open interval $(0, a)$ and, moreover, the integral $\int_0^\infty e^{-\rho} f^2(\rho)d\rho$ is a finite quantity, then the following series (10) with coefficients (11) converges and its sum equals $f(\rho)$ at any point ρ, where this function is continuous [29]:

$$f(\rho) = \sum_{k=0}^{\infty} p_k L_k(\rho), \quad 0 < \rho < \infty, \tag{10}$$

where p_k are defined as follows:

$$p_k = \int_0^\infty f(\rho)e^{-\rho} L_k(\rho)\, d\rho. \tag{11}$$

For particular values of the cone angle 2α and the location of a Stokeslet, i.e., the point $(0, 0, c)$, one can be convinced that functions of the Stokeslet's velocities at the boundary of a cone $v_{0\rho}(\rho,\alpha)$ and $v_{0\theta}(\rho,\alpha)$ are continuous and are satisfied by the conditions of the aforementioned expansion theorem; thus, the decomposition by Laguerre's polynomials occurs and has the following form:

$$v_{0\rho}(\rho,\alpha) = \sum_{n=0}^{\infty} a_n L_n(\rho), \quad v_{0\theta}(\rho,\alpha) = \sum_{n=0}^{\infty} b_n L_n(\rho).$$

We also make a decomposition by Laguerre's polynomials of the constituent part of the velocity's components $\tilde{v}_\rho(\rho,\alpha)$ and $\tilde{v}_\theta(\rho,\alpha)$ at the boundary,

$$\tilde{v}_\rho(\rho,\alpha) = -A_2 P_1(\cos\alpha) - \sum_{n=1}^{\infty} (A_{n+2}P_{n+1}(\cos\alpha) + C_n P_{n-1}(\cos\alpha))\,\rho^n,$$

$$\tilde{v}_\theta(\rho,\alpha) = \frac{1}{\sin\alpha} \sum_{n=0}^{\infty} (n+2)\,(A_{n+2}J_{n+2}(\cos\alpha) + C_n J_n(\cos\alpha))\,\rho^n.$$

For that we use the prominent decomposition of a power function by Laguerre's polynomials. Particularly, if the index s is a nonnegative integer, then the series contains a finite number of summands,

$$\rho^s = s! \sum_{n=0}^{s} (-1)^n \binom{s}{n} L_n(\rho), \quad 0 < \rho < \infty;\ s = 0, 1, 2, \ldots.$$

For mathematical calculations, we make a truncation of the series and consider finite sums of m terms instead of our series. Then, we get the decomposition of Stokeslet's velocities $v_{0\rho}$ and $v_{0\theta}$,

$$v_{0\rho}(\rho,\alpha) = \sum_{n=0}^{m} a_n L_n(\rho), \quad v_{0\theta}(\rho,\alpha) = \sum_{n=0}^{m} b_n L_n(\rho),$$

and the decomposition of additional velocities components $\tilde{v}_\rho$ and $\tilde{v}_\theta$:

$$\tilde{v}_\rho(\rho,\alpha) = -\left(A_2 P_1(\cos\alpha) + \sum_{n=1}^{m}(A_{n+2}P_{n+1}(\cos\alpha) + C_n P_{n-1}(\cos\alpha)n!\right)L_0(\rho) + \sum_{k=1}^{m}\left((-1)^{k+1}\sum_{n=k}^{m}(A_{n+2}P_{n+1}(\cos\alpha) + C_n P_{n-1}(\cos\alpha))n!\binom{n}{k}\right)L_k(\rho),$$

$$\tilde{v}_\theta(\rho,\alpha) = \frac{1}{\sin\alpha}\left(\left(\sum_{n=0}^{m}(n+2)(A_{n+2}J_{n+2}(\cos\alpha) + C_n J_n(\cos\alpha))n!\right)L_0(\rho) + \sum_{k=1}^{m}\left((-1)^k\sum_{n=k}^{m}(n+2)(A_{n+2}J_{n+2}(\cos\alpha) + C_n J_n(\cos\alpha))n!\binom{n}{k}\right)L_k(\rho)\right).$$

After some tedious algebra, we get the linear algebraic systems for finding the corresponding constants.

In the following, we omit the argument $\cos\alpha$ of the Legendre and the Gegenbauer functions for the sake of brevity. The final system of linear algebraic equations to find coefficients C_m and A_{m+2} is as follows:

$$A_{m+2}P_{m+1} + C_m P_{m-1} = \frac{(-1)^m a_m}{m!},$$
$$A_{m+2}J_{m+2} + C_m J_m = \frac{(-1)^{m+1}\sin\alpha\, b_m}{m!(m+2)}.$$

Subsequent coefficients can be found sequentially, using already calculated ones. Namely, for any $k = \overline{1, m-1}$

$$A_{k+2}P_{k+1} + C_k P_{k-1} = \frac{(-1)^k a_k - \sum_{n=k+1}^{m}(A_{n+2}P_{n+1} + C_n P_{n-1})\, n!\binom{n}{k}}{k!},$$
$$A_{k+2}J_{k+2} + C_k J_k = \frac{(-1)^{k+1}\sin\alpha\, b_k - \sum_{n=k+1}^{m}(n+2)(A_{n+2}J_{n+2} + +C_n J_n)n!\binom{n}{k}}{k!(k+2)}.$$

The last few coefficients are unambiguously defined via all previous ones,

$$A_0 = A_1 = 0,\ A_2 = \frac{a_0 - \sum_{n=1}^{m}(A_{n+2}P_{n+1} + C_n P_{n-1})\, n!}{\cos\alpha},$$
$$C_0 = -\frac{\sin\alpha\, b_0 + \sum_{n=1}^{m}(n+2)(A_{n+2}J_{n+2} + C_n J_n)n!}{2} - A_2 J_2.$$

3 Results and discussion

The obtained analytical results allow us to investigate the structure of the flow. We consider an axisymmetric flow inside a cone due to a Stokeslet. Calculations of the streamlines show that the flow has an eddy structure. This cellular character is typical for the Stokes flows. Moffatt [30] was the first one who showed the sequence of eddies near a sharp corner. Later, a similar flow structure was observed by many authors, for example [21,25,31]. As for the cone flow, its description in various cases was given.

As for the details of our computations, we use a truncation with $m = 40$. Our approach allows us to obtain the stream function in a bounded domain with respect to ρ (the domain of series convergence). Pictures of streamlines for these domains are shown in Fig. 2, distinguishing from each other by an opening angle of the cone. The received results are in accordance with those obtained by [13,14,32]. One can note that the intensities of eddies decrease substantially moving away from the apex. The picture of the flow depends on the chosen type of a Stokeslet. We chose the Stokeslet, which is not a source of mass, but a source of momentum.

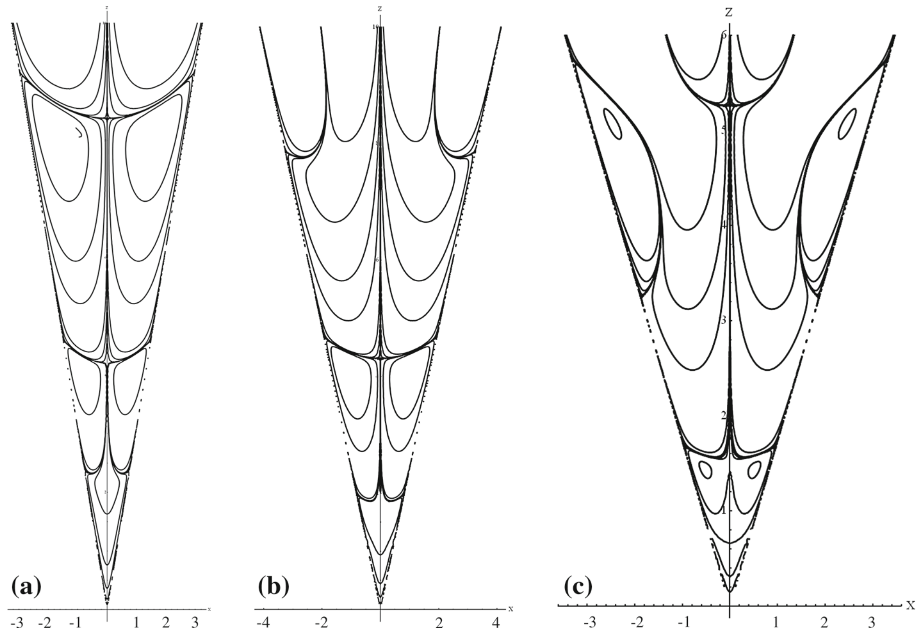

Fig. 2 Streamlines for a flow driven by the Stokeslet at point (0, 0, 4) in the cone with a half-angle: **a** $\alpha = \frac{\pi}{10}$, **b** $\alpha = \frac{\pi}{8}$, **c** $\alpha = \frac{\pi}{6}$

Hence, we have conservation of mass in the fixed volume. The cellular structure is natural for the Stokes flow. In our case, for small opening angles, one can see a vertical (along the cone axis) cellular structure in the domain close to the Stokeslet (Fig. 2a, b). For larger values of the angle, there appears a horizontal cellular structure (Fig. 2c).

The cellular structure of the flow can find interesting chemical applications. One of the intriguing problems of modern nanochemistry is the creation of chemical nanoreactors. The nanoreactor is such a nanostructure that gives us a possibility to perform some chemical reaction strictly inside some small (nanosized) spatial region. A possible way to ensure this result is to collect all reagents for given chemical reaction only in the chosen spatial region. It can be done using mechanical properties. Namely, let the fluid be multicomponent (e.g., it contains different reagents). The existence of an eddy leads to spatial separation of the components (due to the difference of its densities). As a result, one can get the collection of reagents inside this eddy (i.e., in this nanosized region). The separation processes of such types were observed experimentally (see, e.g., [4,33]). To construct the nanoreactor of such kind, it is necessary to know the cellular structure of the flow. The Stokeslet (causing the flow) inside the nanocone can be excited by an external field. Our model gives an instrument for the flow structure description.

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